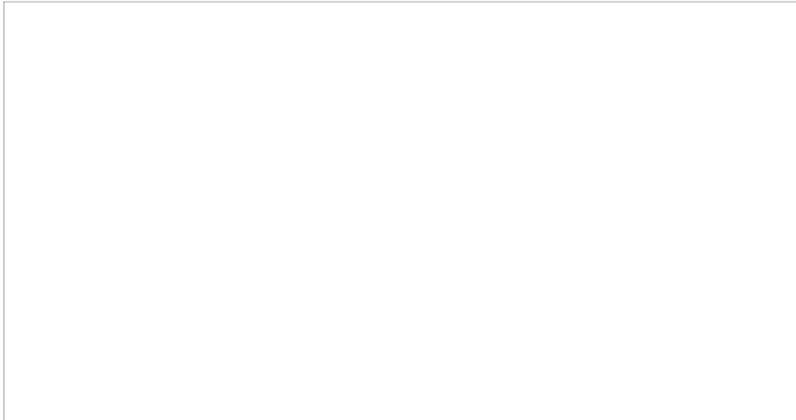


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The Flow of Ground Waters Under Conditions of Locally Accelerated
Infiltration

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THE FLOW OF GROUND WATERS UNDER CONDITIONS OF
LOCALLY ACCELERATED INFILTRATION

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It was shown in (1) that the basic equation expressing the motion of free ground waters under conditions of horizontal water resistance may be linearized and reduced to the equation of thermodynamics

$$a^2 \Delta u + b = \partial u / \partial t \quad (1)$$

$$u = r^2/2, \quad a = \sqrt{k h_{cp} / m}, \quad b = \epsilon h_{cp} / m \quad (2)$$

where h is the depth (pressure) of the ground water, $t = \text{time}$, $k =$ the filtration coefficient of the ground, $m =$ the porosity value of the ground, $\epsilon =$ modulus of the underground run-off, $h_{cp} =$ certain average depth of the ground water current. [In the 7th section of "Hydraulics" in the collection of papers on the theory of ground waters (1959).]

Let us consider one particular solution of equation (1), conformant to the case where the ground current (Figure 1), confined by some area of run-off (for instance, by a river), has a modulus of underground run-off ϵ_0 for sections ab and cd , and a modulus $\epsilon_1 > \epsilon_0$ for section bc . In other words, we shall consider that the ground current, at its section bc , is fed additionally on account of the infiltration from above which has an intensity equal to $\epsilon_1 - \epsilon_0$. Such a case is of interest in the evaluation of the buttressing effect received by ground waters as a result of artificial irrigation and other meliorative measures, which induce locally accelerated infiltration over the part of the [ground] current having a length

$$l = l_2 - l_1 \quad (\text{See Figure 1 in the Appendix}).$$

*[Note: This adjective corresponds to a very common Russian one used to refer to "land improvement", "soil conservation", etc; "meliorative" means literally in Latin "improved, better".]

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Below is a mathematical analysis of the ~~non-steady~~ ^{steady-state} and ~~unsteady~~ ^{nonstationary} ground water flow developing under such conditions.

Steady-State

1. ~~Steady~~ Flow

A. Initially ~~unsteady~~ ^{steady} flow (time $t=0$). The infiltration taking place along the entire current, under natural conditions, is of an intensity $\epsilon = \epsilon_0$.

By interpolation, in equation (1) of the value ϵ_0 in place of ϵ , transposition, and integration, we obtain:

$$\mu = -\frac{1}{2} \frac{q_0}{k} x^2 + Bx + C \quad (3)$$

Substituting into equation (3) the value of the boundary conditions for the area of run-off ~~at~~ at:

$$\mu = f_1/2 = \text{const.} \quad (4)$$

$$g|_{x=0} = \text{const.} = k \frac{d\mu}{dx}|_{x=0} = \text{const.} \quad (5)$$

we find the constants B and C. By ~~inserting~~ ^{inserting} their values into equation (3), we finally obtain:

$$\mu = \frac{f_1^2}{2} + \frac{q_0}{2} x - \frac{1}{2} \frac{q_0}{k} x^2 \quad (6)$$

where $q_0 =$ discharge [volume] of ground waters flowing into the area of the run-off. Knowing the level of the ground waters under natural conditions at some point $x = x_1$, it is possible to determine from equation (6) the value of q_0 , which, therefore, is to be considered as known.

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B. Finally ^{steady} flow (time $t = \infty$). For ~~limit~~ ^{limit *} conditions, when $t = \infty$, the value of ξ in equation (1) will be, for section bc, $\xi = \xi_1$, and for sections ab and cd, $\xi = \xi_0$. ^{*"going to the limit"}
~~Inserting~~ ^{Inserting} into equation (1) these values of ξ , assuming that $\partial h / \partial t = 0$, and integrating equation (1), we obtain:

$$ab: u_1 = -\frac{1}{2} \frac{K}{L} x^2 + B_1 x + C_1 \quad (7)$$

$$bc: u_2 = -\frac{1}{2} \frac{K}{L} x^2 + B_2 x + C_2 \quad (8)$$

$$cd: u_3 = -\frac{1}{2} \frac{K}{L} x^2 + B_3 x + C_3 \quad (9)$$

Let's assume that ~~limit~~ ^{limit} conditions at the area of run-off are as controlled by equations:

$$h|_{x=0} = h^0/2 \quad (10)$$

$$q|_{x=0} = q_0, \quad h|_{x=L} = q_0 t / K \quad (11)$$

and also that ~~limit~~ ^{limit} conditions at the boundaries of sections with different infiltration values bf and cg, are as controlled by equation:

$$u|_{x=0} = u|_{x=L}, \quad q|_{x=0} = q|_{x=L}, \quad u|_{x=L} = u|_{x=0}, \quad q|_{x=L} = q|_{x=0} \quad (12)$$

Condition (11) means that the inflow of water from the feeding area, after the genesis of accelerated infiltration, remains unchanged; and therefore the discharge volume of the ground waters

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which flows into the area of run-off changes only on account of the additional infiltration $\mathcal{E}_1 = \mathcal{E}_a$ taking place at section, the length of which is expressed by l_1 .

Substituting conditions (10), (11), (12) into equations (7), (8), (9), we obtain constants $B_1, B_2, B_3, C_1, C_2, C_3$.

By ~~substituting~~ ^{inserting} the values of these constants into equation (1), we finally obtain:

$$a. \quad u_1 = \frac{q_1}{2} - \frac{1}{2} \frac{q_1}{K} x^2 + \left(\frac{q_1}{K} + \frac{q_1}{K} \right) x. \quad (13)$$

$$b. \quad u_2 = \frac{q_2}{2} - \frac{1}{2} \frac{q_2}{K} x^2 + \left(\frac{q_2}{K} + \frac{q_2}{K} l \right) x - \frac{1}{2} \frac{q_2}{K} l^2. \quad (14)$$

$$c. \quad u_3 = \frac{q_3}{2} - \frac{1}{2} \frac{q_3}{K} x^2 + \frac{q_3}{2} l - \frac{1}{2} \frac{q_3}{K} \left(l^2 - x^2 \right). \quad (15)$$

2. ~~Steady~~ ^{Nonstationary} Flow

In investigating an ~~unsteady~~ ^{unsteady} flow, we shall assume that the additional infiltration into section bc is generated instantly, at the moment of time $t = 0$, and is subsequently maintained as a constant value.

With this limitation, the problem is posed in the following manner: it is required to obtain ^{the} function $u(x, t)$ which, in the interval $l_1 < x < l_2$, satisfies equation (1), with $\mathcal{E} = \mathcal{E}_1$, and initial condition (6); and, in the intervals $x < l_1$, and $x > l_2$, it satisfies equation (1), with $\mathcal{E} = \mathcal{E}_0$ and initial

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condition (6); and, with $x = 0$, it satisfies the boundary condition (4).

In the case where $t = \infty$, the flow under investigation is determined by equations (13), (14), (15).

A similar ~~mathematical~~ problem is posed in thermodynamics, in the study of the heating of metal during the process of electric welding⁽²⁾, and also in the theory of filtration⁽³⁾. In our present problem, the initial conditions are stated in equation (6), while, in ~~electric welding~~⁽²⁾, the initial temperature is considered as equaling zero. Additionally, the area of run-off (discharge) is ~~inserted~~ to our given problem, as a result of which the ~~unsteady~~ flow approaches a state of dynamic equilibrium, as determined by equations (13), (14), (15). With relation to the problem posed by electric welding⁽²⁾, the ~~unsteady~~ movement of heat also approximates asymptotically some static thermal field, but this takes place at the expense of external cooling. With relation to the ~~theory~~ of filtration⁽³⁾, the ~~unsteady~~ filtration cycle progresses toward the ~~limit~~ flow as a result of the water percolating through the ~~water-tight region~~ low-permeability ~~medium~~. If, in both cases^(2,3), the above factors are overlooked, the temperature (or pressure) will keep increasing indefinitely with time, progressing toward infinity.

For the solution of the problem in hand, as in case⁽²⁾, it is convenient to utilize the so-called "thermal impact" function, which satisfies equation (1). In the absence of external cooling, this function is expressed, as follows:

$$f(x,t) = \frac{e^{-x^2/4a^2t}}{2a\sqrt{\pi t}} \quad (16)$$

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...in the limit,
 effecting the summation of the individual "thermal
 impacts," and utilizing, in addition, the principle of superposition,
 we obtain the solution of the problem, as follows:

$$u = A_0 \int_0^t \int_{-l_1}^{l_2} \frac{1}{2a\sqrt{t-\theta}} \left[e^{-\frac{(x-l_0-\xi)^2}{4a^2(t-\theta)}} - e^{-\frac{(x+l_0-\xi)^2}{4a^2(t-\theta)}} \right] d\theta d\xi + B_1 x^2 + C_1 x + D_1 =$$

$$= \frac{A_0}{\sqrt{\pi}} \int_0^t \left[\int_{-l_1}^{l_2} e^{-\alpha^2} d\alpha - \int_{-l_2}^{l_1} e^{-\beta^2} d\beta \right] d\theta + B_1 x^2 + C_1 x + D_1 \quad (17)$$

where

$$l_1 = l_0 + \frac{1}{2} \sqrt{\frac{4a^2 t - l_0^2}{t}}, \quad \alpha = \frac{x-l_0-\xi}{\sqrt{t}}, \quad \beta = \frac{x+l_0-\xi}{\sqrt{t}} \quad (18)$$

$$\alpha_1 = \frac{x-l_1}{\sqrt{t}}, \quad \alpha_2 = \frac{x-l_2}{\sqrt{t}}, \quad \beta_1 = \frac{x+l_1}{\sqrt{t}}, \quad \beta_2 = \frac{x+l_2}{\sqrt{t}} \quad (19)$$

while

$$T = 2a\sqrt{t-\theta} \quad (20)$$

By applying to equation (17) the method of least squares, and determining the values of the constants A, B, C, D by equation (1), with $\xi = \xi_1$, also by boundary condition (h) and initial condition (6), we obtain:

$$a.b: \quad u = \frac{f_1^2}{2} - \frac{1}{2} \frac{\xi_0}{k} x^2 + \left(\frac{q_0}{k} + \frac{\xi_1 - \xi_0}{k} \right) x + \frac{1}{4} \frac{\xi_1 - \xi_0}{k} x^2 S \quad (21)$$

$$b.c: \quad u = \frac{f_1}{2} - \frac{1}{2} \frac{\xi_1 - \xi_0}{k} x^2 - \frac{1}{2} \frac{\xi_1}{k} x^2 + \left(\frac{q_0}{k} + \frac{\xi_1 - \xi_0}{k} l_1 \right) x + \frac{1}{4} \frac{\xi_1 - \xi_0}{k} x^2 S, \quad (22)$$

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$$\begin{aligned} \text{cd: } \mu &= \frac{p^2}{2} + \frac{1}{2} \frac{\epsilon_1 - \epsilon_0}{\lambda} (\ell_2^2 - \ell_1^2) - \frac{1}{2} \frac{\epsilon_0}{\lambda} x^2 + \frac{z}{\lambda} + \\ &+ \frac{1}{4} \frac{\epsilon_1 - \epsilon_0}{\lambda} z^2 S, \end{aligned} \quad (23)$$

where

$$\begin{aligned} S &= \left(1 - \frac{\ell_1}{z}\right)^2 F(\lambda_1) - \left(1 - \frac{\ell_2}{z}\right)^2 F(\lambda_2) - \left(1 + \frac{\ell_3}{z}\right)^2 F(\lambda_3) + \\ &+ \left(1 + \frac{\ell_4}{z}\right)^2 F(\lambda_4), \end{aligned} \quad (24)$$

while

$$F = \left(1 + \frac{1}{2\lambda^2}\right) \Phi(\lambda) + \frac{1}{\sqrt{\pi}} \frac{1}{\lambda} e^{-\lambda^2} \quad (25)$$

$$\lambda_1 = \frac{x - \ell_1}{2\lambda_1 t}, \quad \lambda_2 = \frac{x - \ell_2}{2\lambda_2 t}, \quad \lambda_3 = \frac{x + \ell_3}{2\lambda_3 t}, \quad \lambda_4 = \frac{x + \ell_4}{2\lambda_4 t} \quad (26)$$

$$\Phi(\lambda) = \frac{2}{\sqrt{\pi}} \int_0^\lambda e^{-\lambda^2} d\lambda. \quad (27)$$

Bearing in mind, that $\phi(-\lambda) = -\phi(\lambda)$, $\phi(0) = 0$, and $\phi(\infty) = 1$ we obtain from equation (25), as follows:

$$\begin{aligned} F(-\lambda) &= -F(\lambda), \quad F(\lambda)|_{t=\infty} = \infty, \quad F(\lambda_1)|_{t=0, x < \ell_1} = -1, \\ F(\lambda_1)|_{t=0, x > \ell_1} &= 1, \quad F(\lambda_2)|_{t=0, x < \ell_2} = -1, \quad F(\lambda_2)|_{t=0, x > \ell_2} = 1, \quad F(\lambda_3)|_{t=0} = 1, \\ F(\lambda_4)|_{t=0} &= 1. \end{aligned} \quad (28)$$

Subsequently, from equation (24), we determine:

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$$S|_{x=0} = 0, \quad S|_{t=\infty} = 0, \quad S|_{t=c, x < l_1} = -\frac{4(l_2 - l_1)}{x}$$

$$S|_{t=c, l_1 < x < l_2} = \frac{2(x^2 - 2cl + l^2)}{x^2}, \quad S|_{t=c, x > l_2} = \frac{2(l_2^2 - l_1^2)}{x^2} \quad (29)$$

Taking into account these peculiar properties of the functions \bar{h} and S , it becomes a simple matter to check that equations (21), (22), (23), when $t = 0$, can be converted into equation (6); and when $t = \infty$, they can be converted into equations (13), (14), (15). When $x = 0$, equation (21) becomes equation (1). Equations (21), (22), (23) also satisfy the conditions at the boundaries of sections with different infiltration values, as controlled by equation (12). The solution of equations (21), (22), (23) can also be generalized to include the case of an increase in the water level of a river from h_1 to h_2 , which occurs simultaneously with the genesis of locally accelerated infiltration. To accomplish this, it is necessary to add the term $-\frac{1}{2}(h_2^2 - h_1^2) \Phi\left(\frac{x}{2a\sqrt{t}}\right)$ to the right-hand term of equations (21), (22), (23), and substitute the value of h_1 by the value of h_2 .

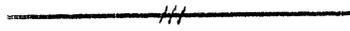
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APPENDIX



Figure 1.

— END —

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